GLG362/GLG598 Geomorphology
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Evolution of Depositional Alluvial River Profiles

A. Recap: Essentials from Flow Mechanics and Sediment Transport

1. Conservation of Momentum, Steady Uniform Flow

$$\tau_b = \rho g h S \qquad S = -\frac{dz}{dx}$$

$$\tau_b = \rho C_f \bar{u}^2 \qquad \frac{\bar{u}}{u_*} = C_f^{-1/2} \qquad u_* = \sqrt{\frac{\tau_b}{\rho}}$$

Note: Unsteady: $\frac{\partial \bar{u}}{\partial t}$ is usually small; exceptions: dam break floods, etc.

Non-Uniform: $\bar{u}\frac{\partial \bar{u}}{\partial x}$; $\rho g \frac{\partial h}{\partial x}$ -- important in flow around bends, over point bars, across abrupt changes in channel slope or width

2. Velocity Profiles

$$u(z) = \frac{u_*}{k}\ln\frac{z}{z_o}$$

$$\langle u \rangle = \frac{u_*}{k}\left(\ln\frac{h}{z_o} - 1\right) = u\big|_{z=.37h}$$

3. Sediment Transport

Shield's Stress

$$\tau_* = \frac{\tau_b}{(\rho_s - \rho) g D}$$

steady-uniform flow:

$$\tau_* = \frac{hS}{((\rho_s - \rho)/\rho)D}$$

Critical Shear Stress for Initiation of Motion

Shield's diagram $\tau_{*cr} = f(R_{ep})$
(explicit particle Reynold's number)

For Gravel: $\tau_{*cr} \approx 0.03 - 0.06$

Non-Dimensional Sediment Transport

$$q_s = \frac{Q_s}{w}$$

$$q_{s*} = \frac{q_s}{\sqrt{((\rho_s - \rho)/\rho)gD}D} = \frac{q_s}{R_{ep}\nu}$$

Bedload Transport (gravel)

$$q_{s*} = 8(\tau_* - \tau_{cr*})^{3/2}$$

$$q_s = \frac{8}{\rho^{3/2}((\rho_s - \rho)/\rho)g}(\tau - \tau_{cr})^{3/2}$$

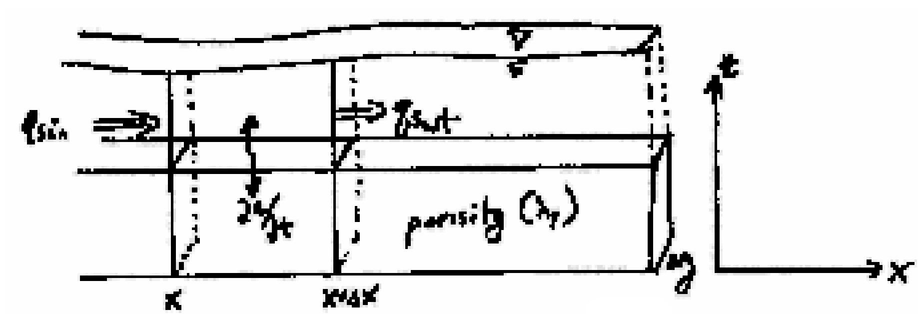
Total Load Sand Transport

$$q_{s*} = \frac{0.05}{C_f}\tau_*^{2.5}$$

B. Exner Equation (Erosion Equation): Conservation of Mass (Sediment)

Derivation of conservation of mass – the erosion equation

SKETCH: Control reach, width Δy, length Δx, q_{s_in} at x, q_{s_out} at $x + \Delta x$



If more sediment comes in than out, bed elevation goes up – deposition
If more sediment goes out than in, bed elevation goes down – erosion

per unit time Δt, sediment volume in = $q_{s\_in}\Delta t\Delta y$
per unit time Δt, sediment volume out = $q_{s\_out}\Delta t\Delta y$

Change in sediment volume per unit time

$$\Delta V_s = q_{s\_in}\Delta t \Delta y - q_{s\_out}\Delta t \Delta y$$

Change in bed volume per unit time

$$\Delta V_{bed} = \Delta x \Delta y \Delta z = \frac{\Delta V_s}{1-\lambda_p} = \frac{q_{s\_in}\Delta t \Delta y - q_{s\_out}\Delta t \Delta y}{1-\lambda_p}$$

For change in bed elevation, divide through by $\Delta x \Delta y \Delta t$:

$$\frac{\Delta z}{\Delta t} = \frac{1}{(1-\lambda_p)}\left(\frac{q_{s\_in} - q_{s\_out}}{\Delta x}\right) = -\frac{1}{(1-\lambda_p)}\left(\frac{\Delta q_s}{\Delta x}\right)$$

$$\frac{\partial z}{\partial t} = -\frac{1}{(1-\lambda_p)}\frac{\partial q_s}{\partial x}$$

Note from basics of sediment transport that $q_s = f(\tau_b)$
Accordingly, we can expand the erosion equation using the chain rule:

$$\frac{\partial z}{\partial t} = -\frac{1}{(1-\lambda_p)}\frac{\partial q_s}{\partial \tau_b}\frac{\partial \tau_b}{\partial x}$$

Thus we can see immediately that erosion will occur in regions of *increasing* shear stress (i.e., not necessarily in regions of *high* shear stress) and deposition will occur in regions of *decreasing* shear stress (not necessarily *low* shear stress).

C. Channel Width Closure

Problem: all relations above derived in terms of flow depth, shear stress, discharge per unit width, but channel width changes downstream with: changing Q_w, changing slope (S), changing D_{50}, changing vegetation, *etc*.:
To solve problem of alluvial river profile evolution we must specify how channel width evolves downstream.

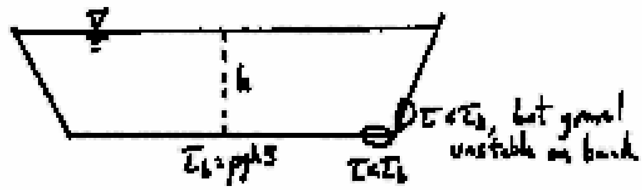
1. Hydraulic Geometry (Leopold et al, 1950s)

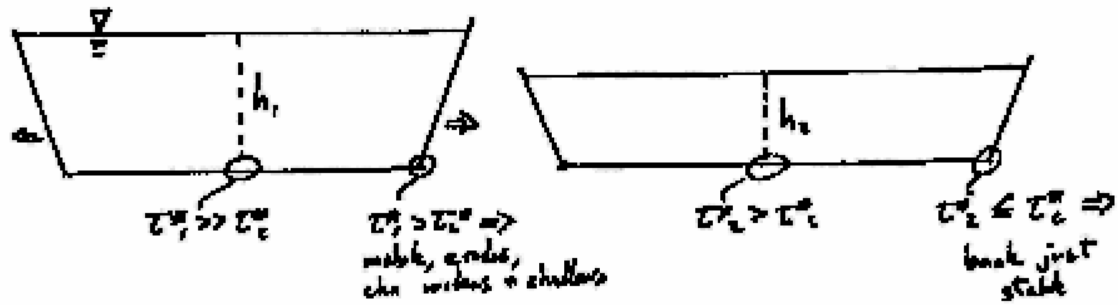
Empirical: $w \propto Q^{0.5}$

2. Equilibrium (Threshold) Straight, Gravel-bed Channels (Parker, 1978)

Concept: channel will widen until banks are just stable, just below the threshold for motion (erosion = widening)

SKETCH





Critical or Equilibrium Condition: $\tau_* = (1+\varepsilon)\tau_{*c}$; $\varepsilon = 0.2 - 0.4$

Summary Condition at Bankfull flow: Mobile bed, stable banks, generally low transport stage

Thus, for given Q_w, D_{50}, S, C_f, Width (w) increases downstream such that h reduces to establish $\tau_* = (1+\varepsilon)\tau_{*c}$

3. Sandy, Meandering Channels

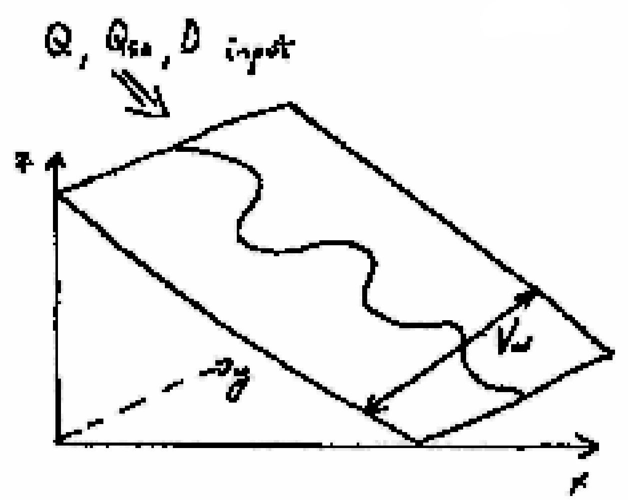
Theory not well developed, but some evidence indicates near constant shield's stress:

$$\tau_* = (1+\varepsilon)\tau_{*c} \; ; \quad \varepsilon = 1.2 - 1.4$$

Otherwise, generally the assumption that $\tau_* >> \tau_{*c}$ is often reasonably accurate.

D. Relations for Alluvial Plain Slope (Paola, 1992; with corrections)

DEFINITION SKETCH



1. Conservation of Mass (Water)

$$Q = \bar{u}hw$$

$$\frac{Q}{V_w} = q_v = \bar{u}h\frac{w}{V_w} = \bar{u}h\beta \quad ; \quad \beta \equiv \frac{w}{V_w}$$

Note q_v denotes water discharge per unit valley width, as opposed to q which we have used previously for water discharge per unit channel width.

Parallel drainage: V_w = constant; no lateral inflows of water or sediment, no loss of water to infiltration or evaporation.

$$\frac{\partial Q}{\partial x} = 0$$

2. Conservation of Mass (Sediment)

$$Q_S = q_s w$$

$$\frac{Q_s}{V_w} = q_{sv} = q_s \frac{w}{V_w} = q_s \beta$$

where q_{sv} is sediment transport per unit valley width [m^2/s]

$$\frac{\partial z}{\partial t} = -\frac{1}{1-\lambda_p}\frac{\partial q_{sv}}{\partial x}$$

3. Conservation of Momentum

$$\tau_b = \rho g h S \qquad S = -\frac{dz}{dx}$$

$$\tau_b = \rho C_f \bar{u}^2 \qquad C_f^{-1/2} = \frac{\bar{u}}{u_*}$$

4. Sediment Transport (Bedload = gravel)

$$q_{s*} = \frac{q_s}{\sqrt{((\rho_s - \rho)/\rho) gD}D} = 8(\tau_* - \tau_{cr*})^{3/2}$$

Dimensional sediment flux per unit channel width:

$$q_s = 8\sqrt{((\rho_s - \rho)/\rho) gD}D\left(\frac{\tau - \tau_{cr}}{(\rho_s - \rho)gD}\right)^{3/2}$$

$$q_s = \frac{8}{\rho^{3/2}((\rho_s - \rho)/\rho)g}(\tau - \tau_{cr})^{3/2}$$

5. Channel width closure:

Braided, gravel-bedded channel --

$$\tau = (1+\varepsilon)\tau_{cr}$$

$$\tau - \tau_{cr} = \tau\left(\frac{\varepsilon}{1+\varepsilon}\right)$$

Meandering, sand-bedded channel –

$$\tau - \tau_{cr} = \tau$$

6. Relation for sediment flux using channel closure rule(s)

$$q_s = \frac{8c_w}{\rho^{3/2}((\rho_s - \rho)/\rho)g}\tau^{3/2}$$

$$c_w = \left(\frac{\varepsilon}{1+\varepsilon}\right)^{\frac{3}{2}}$$ Braided, gravel-bedded channels

$$c_w = 1$$ Meandering, sand-bedded channels

Write this relation in terms of slope and water discharge per unit valley width, q_v

$$\tau^{3/2} = \tau\tau^{1/2}$$

Using two relations for conservation of momentum:

$$\tau^{3/2} = -\rho g h \frac{\partial z}{\partial x}\left(\rho C_f \bar{u}^2\right)^{1/2}$$

Using conservation of mass for water:

$$\tau^{3/2} = -\rho^{3/2} g \sqrt{C_f}\, h\bar{u}\frac{\partial z}{\partial x} = -\rho^{3/2} g \sqrt{C_f}\,\frac{q_v}{\beta}\frac{\partial z}{\partial x}$$

Substitute into relation for sediment flux (per unit channel width):

$$q_s = -\frac{8 c_w q_v \sqrt{C_f}}{\beta\left(\left(\rho_s - \rho\right)/\rho\right)}\frac{\partial z}{\partial x}$$

$$q_s = -K_f \frac{\partial z}{\partial x} \quad ; \quad K_f = \frac{8 c_w q_v \sqrt{C_f}}{\beta\left(\left(\rho_s - \rho\right)/\rho\right)} \quad \text{(fluvial transport coefficient)}$$

No dependence on grainsize? Why? -- Channel width closure accounts for grainsize, ie. channel width adjusts according to grainsize.

Note q_s is sediment transport per unit channel width.

7. Conservation of Mass (sediment) (from above)

$$\frac{\partial z}{\partial t} = -\frac{1}{1-\lambda_p}\frac{\partial q_{sv}}{\partial x}$$

recall q_{sv} is sediment transport per unit valley width

$$q_{sv} = \beta q_s = -\beta K_f \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial t} = \frac{1}{1-\lambda_p}\frac{\partial}{\partial x}\left(\beta K_f \frac{\partial z}{\partial x}\right)$$

Assume: constant V_w, C_f, q (and note β is not constant, but cancels out)

$$\frac{\partial z}{\partial t} = \frac{\beta K_f}{1-\lambda_p}\frac{\partial^2 z}{\partial x^2} \qquad \text{(diffusion equation)}$$

Effective Diffusivity:

$$D_f = \frac{\beta K_f}{1-\lambda_p} = \frac{8c_w \rho q_v \sqrt{C_f}}{(\rho_s - \rho)(1-\lambda_p)}$$
$$\frac{\partial z}{\partial t} = D_f \frac{\partial^2 z}{\partial x^2}$$

Sediment inflow (delivered from upstream erosional source area) is Q_{so}.

Steady-state condition:

$$\frac{\partial z}{\partial t} = 0 = \frac{\partial q_{sv}}{\partial x}$$

Thus sediment flux per unit valley width q_{sv} is constant at steady state:

$$q_{sv}(x) = q_{sv}\big|_{x=0} = \frac{Q_{so}}{V_w}$$

Whether steady-state or not, slope at inflow must be sufficient to carry all sediment:

$$q_{sv}\big|_{x=0} = \frac{Q_{so}}{V_w} = -\beta K_f \frac{\partial z}{\partial x}\Big|_{x=0}$$

Thus inlet slope must be (under all conditions):

$$\frac{\partial z}{\partial x}\Big|_{x=0} = -\frac{Q_{so}}{V_w \beta K_f} = -\frac{Q_{so}}{wK_f} = -\frac{Q_{so}\beta((\rho_s - \rho)/\rho)}{w8c_w q_v \sqrt{C_f}} = -\left[\frac{(\rho_s - \rho)}{8c_w \rho \sqrt{C_f}}\right]\frac{Q_{so}}{Q}$$

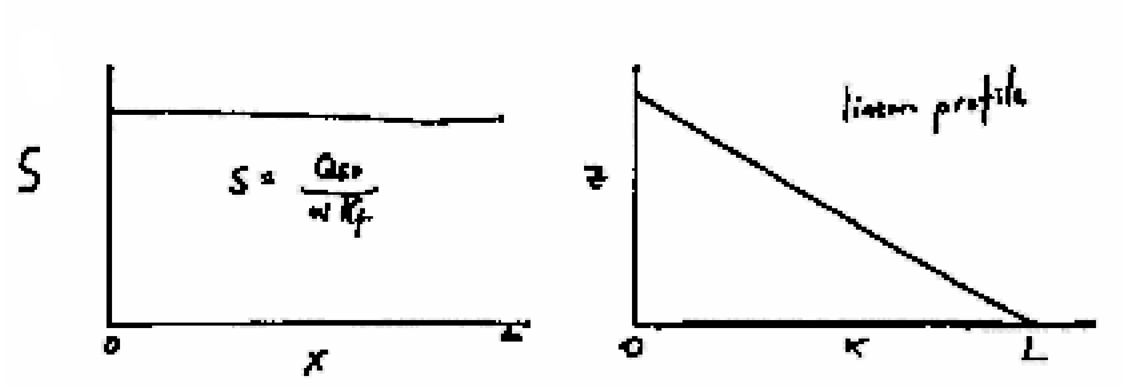
where the term in brackets collects physical constants. Inlet slope is linearly dependent on the ratio Q_{so}/Q. *Note* that valley width and sediment porosity do not influence the inlet slope.

Steady state condition:

$$\frac{\partial z}{\partial t} = 0 = \frac{\beta K_f}{1-\lambda_p}\frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial z}{\partial x} = const = \frac{\partial z}{\partial x}\Big|_{x=0} = -\frac{Q_{so}}{wK_f}$$ (linear profile, slope = inlet slope)

SKETCHES



FROM HERE ON IS EXTRA MATERIAL NOT COVERED IN GLG362

8. Effect of subsidence (σ) on steady-state profile

$$\frac{\partial z}{\partial t} = -\sigma + \frac{\beta K_f}{1-\lambda_p}\frac{\partial^2 z}{\partial x^2}$$

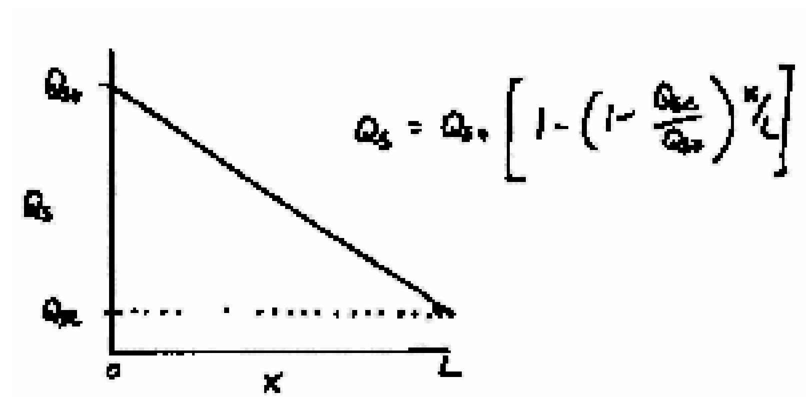
At steady state, the rate of aggradation must balance subsidence, here uniform in space.

Boundary conditions

$$Q_s\big|_{x=0} = Q_{so} \qquad ; \qquad Q_s\big|_{x=L} = Q_{se}$$

Gives

$$Q_s(x) = Q_{so}\left[1-\left(1-\frac{Q_{se}}{Q_{so}}\right)\frac{x}{L}\right]$$; linear decline from Q_{so} to Q_{se}



And since at steady state, deposition must balance subsidence, the total volume rate of deposition (($Q_{so} - Q_{se}$)/(1-λ_p)) divided by alluvial plain area (LV_w) equals the subsidence rate:

$$\sigma = \frac{Q_{so} - Q_{se}}{(1-\lambda_p)V_w L}$$

Another way to think about this is to ask, what sets the length of the alluvial plain (L)? The answer is obtained by solving the above relation for L. Basically the total volume rate of deposition, valley width, and subsidence rate set the length of the alluvial plain, or the distance of gravel progradation into a depositional basin:

$$L = \frac{Q_{so} - Q_{se}}{(1-\lambda_p)V_w \sigma}$$; for the case of gravel progradation, $Q_{se} = 0$ would be the

condition of interest – all gravel is deposited in the gravel wedge.

So we have resolved the controls on the size of the system, and the pattern of $Q_s(x)$ down the system. Now we can ask, 'what sets the system slope and longitudinal profile?' In the case of zero aggradation (same as $Q_{se} = Q_{so}$), the alluvial plain had a constant slope, linear profile. Intuitively, in the case of subsidence balanced by aggradation, will the profile be concave-up or convex-up? Recall that sediment flux declines linearly down the system in this case. For the case of a gravel plain, $Q_{se} = 0$ we have:

$$Q_s(x) = Q_{so}\left[1 - \frac{x}{L}\right] \quad ; \quad \text{with} \quad L = \frac{Q_{so}}{(1-\lambda_p)V_w \sigma}$$

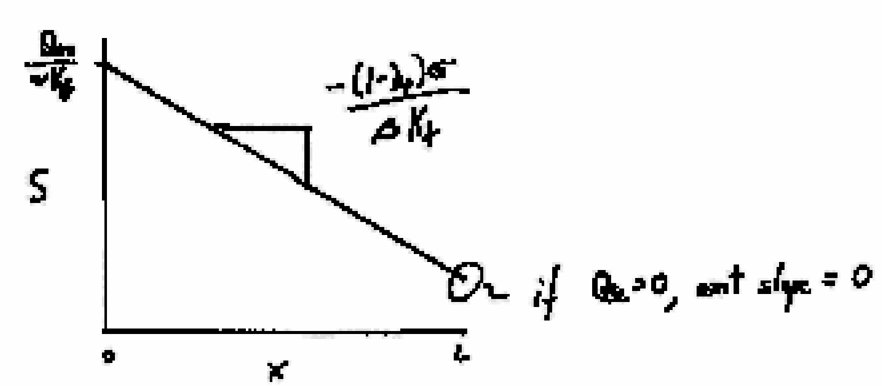
$$Q_s(x) = Q_{so} - x(1-\lambda_p)V_w \sigma$$; which gives $Q_s = 0$ at $x = L$, as required.

From above, we know that locally the slope is linearly dependent on sediment flux:

$$\frac{\partial z}{\partial x} = -\frac{Q_s}{wK_f} = -\left[\frac{(\rho_s - \rho)}{8c_w \rho \sqrt{C_f}}\right]\frac{Q_{so}}{Q}$$; thus substituting the relation for $Q_s(x)$:

$$\frac{\partial z}{\partial x} = -\frac{Q_s}{wK_f} = -\frac{Q_{so} - x(1-\lambda_p)V_w \sigma}{wK_f} = -\frac{Q_s}{wK_f} + \frac{x(1-\lambda_p)\sigma}{\beta K_f}$$

$$-\frac{\partial z}{\partial x} = S = \frac{Q_{so}}{wK_f} - \frac{(1-\lambda_p)\sigma}{\beta K_f}x$$



Naturally, we can find the same result by direct integration of the conservation of mass (or erosion-transport) equation:

Steady state condition

$$\frac{\partial z}{\partial t} = 0 = -\sigma + \frac{\beta K_f}{1-\lambda_p}\frac{\partial^2 z}{\partial x^2} \qquad ; \qquad \sigma = \frac{\beta K_f}{1-\lambda_p}\frac{\partial^2 z}{\partial x^2}$$

Integrate once:

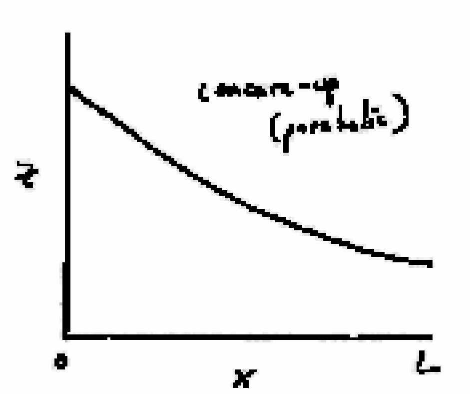
$$\frac{\partial z}{\partial x} = \frac{(1-\lambda_p)\sigma}{\beta K_f}x + const$$

$$const = \frac{\partial z}{\partial x}\Big|_{x=0} = -\frac{Q_{so}}{wK_f}$$

$$\frac{\partial z}{\partial x} = \frac{(1-\lambda_p)\sigma}{\beta K_f}x - \frac{Q_{so}}{wK_f}$$

$$S = -\frac{\partial z}{\partial x} = \frac{Q_{so}}{wK_f} - \frac{(1-\lambda_p)\sigma}{\beta K_f}x$$

Concave-up profile with slope decreasing linearly from the inlet (i.e., a parabola) at a rate determined by the fluvial diffusivity, subsidence rate, sediment porosity, and the channel width to valley width ratio. Note that the limiting case of 100% porosity converges on the zero deposition (subsidence) case, as it must.



9. Effect of Uplift on Steady-State Profile

$$\frac{\partial z}{\partial t} = U + \frac{\beta K_f}{1-\lambda_p}\frac{\partial^2 z}{\partial x^2}$$

Directly analogous, opposite sign on source term, so jump to solution:

$$S = -\frac{\partial z}{\partial x} = \frac{Q_{so}}{wK_f} + \frac{(1-\lambda_p)U}{\beta K_f}x$$

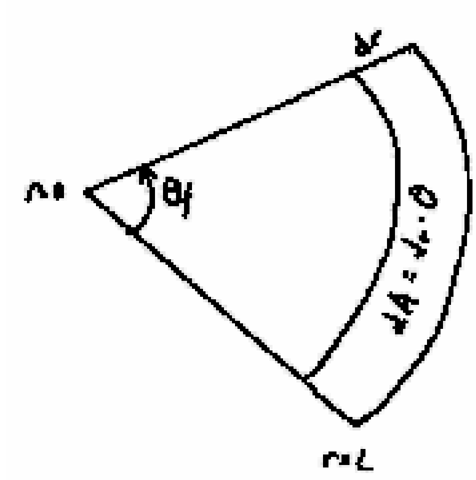
Convex-up profile with slope increasing linearly from the inlet at a rate determined by the fluvial diffusivity, uplift rate, sediment porosity, and the channel width to valley width ratio.

Note this result is unrealistic in that an incising system will dissect the alluvial plain into a dendritic drainage pattern – the result here assumes the channel sweeps back and forth to maintain a one-dimensional form – a tilting plain – as is the case for an aggradational system. We will discuss incising systems later.

10. Adaptation for Alluvial Fans at Steady-state (aggradation = uniform subsidence)

Assume radially symmetrical fan, *r* is radial position (from 0 to *L*), fan is pie-shape with apex angle θ_f.

SKETCH

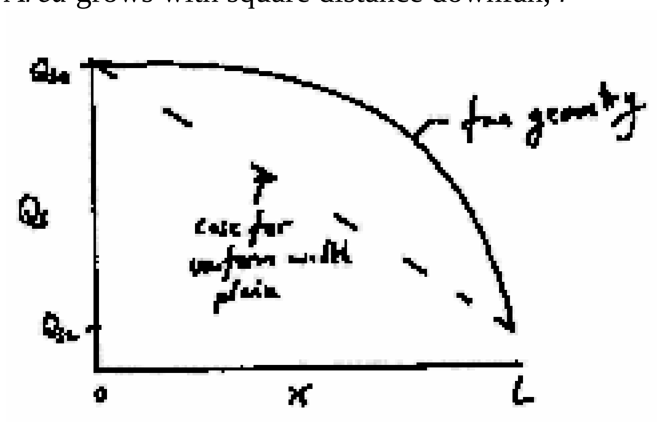


Boundary conditions are:

$$Q_s\big|_{r=0} = Q_{so} \qquad ; \qquad Q_s\big|_{r=L} = Q_{se}$$

Gives

$$Q_s(r) = Q_{so}\left[1 - \left(1 - \frac{Q_{se}}{Q_{so}}\right)\left(\frac{r}{L}\right)^2\right]$$; non-linear decline from Q_{so} to Q_{se} because fan *Area* grows with square distance downfan, r^2



Again, at steady state deposition must balance subsidence, the total volume rate of deposition $((Q_{so} - Q_{se})/(1-\lambda_p))$ divided by alluvial fan area $(\theta_f L^2/2)$ equals the subsidence rate:

$$\sigma = \frac{Q_{so} - Q_{se}}{(1-\lambda_p)\theta_f\left(L^2/2\right)}$$

Solving for steady-state fan size:

$$L = \sqrt{\frac{Q_{so} - Q_{se}}{(1-\lambda_p)\theta_f(\sigma/2)}} \quad .$$

So we have resolved the controls on the size of the system, and the pattern of $Q_s(r)$ down the system, and how these differ due to the expanding geometry of the alluvial fan. Now we can ask, 'what difference does the fan geometry make to the longitudinal profile?' Intuitively, will the profile be more or less concave than the alluvial plain? Recall that sediment flux declines with the square of distance down the fan in this case. For the case of a gravel fan merging with a playa for, for instance, $Q_{se} = 0$ we have:

$$Q_s(r) = Q_{so}\left[1 - \left(\frac{r}{L}\right)^2\right] \quad ; \quad \text{with} \quad L = \sqrt{\frac{Q_{so}}{(1-\lambda_p)\theta_f(\sigma/2)}}$$

$$Q_s(r) = Q_{so} - \frac{(1-\lambda_p)\theta_f\sigma}{2}r^2$$; which gives $Q_s = 0$ at $r = L$, as required.

From above, we know that locally the slope is linearly dependent on sediment flux:

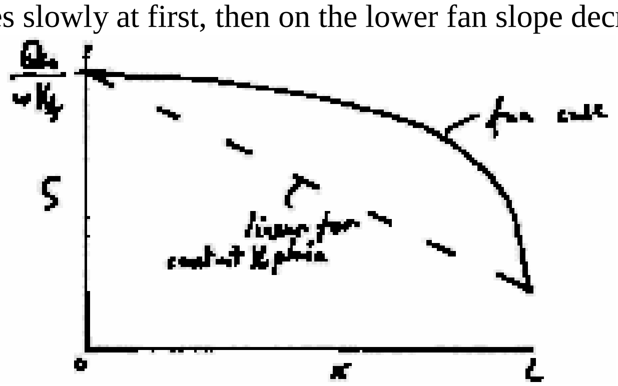
$$\frac{\partial z}{\partial r} = -\frac{Q_s}{wK_f}$$; (no difference from alluvial plain case) thus substituting the

relation for $Q_s(r)$:

$$\frac{\partial z}{\partial r} = -\frac{Q_s(r)}{wK_f} = -\frac{Q_{so} - (1-\lambda_p)\theta_f\,\sigma/2}{wK_f}r^2 = -\frac{Q_s}{wK_f} + \frac{(1-\lambda_p)\theta_f\sigma}{2wK_f}r^2$$

$$-\frac{\partial z}{\partial r} = S = \frac{Q_s}{wK_f} - \frac{(1-\lambda_p)\theta_f\sigma}{2wK_f}r^2$$

Thus, the concavity of alluvial fans is quite distinct from that of alluvial plains: near the fan-head the slope remains close to constant (linear profile) as $Q_s(r)$ decreases slowly at first, then on the lower fan slope decreases rapidly to zero.



11. System Response Time to Perturbation

Response time is well known for Diffusion equation (see Paola et al. 1992)

$$D_f = \frac{\beta K_f}{1-\lambda_p} \quad \text{(Effective Diffusivity)}$$

$$T_{eq} = \frac{L^2}{D_f} \quad ; \quad \text{where } L \text{ is system (alluvial plain) length.}$$

View graphs: System response to slow ($T > T_{eq}$) and fast ($T < T_{eq}$) perturbations (Figures from Paola et al., 1992).