

## Mathematical Description of Processes

### Conservation of Mass

[Sketch]  $h = z \cos \alpha$      $\Delta s = \frac{\Delta x}{\cos \alpha}$

Volume of sediment (V); Volume flux of soil ( $q_v$ ) [x-component]; Depth of soil (h); Elevation soil surface (z);  $\Delta s$ ;  $\Delta x$ ;  $\Delta t$  Note for creep processes volume flux of soil includes the pore space (not volume flux of sediment), for sheetwash can define transport rate of soil or transport rate of sediment; the latter requires a porosity correction in mass balance equation.

Volume of sediment in box (unit width)     $V = h\Delta s = z \cos \alpha \frac{\Delta x}{\cos \alpha} = z\Delta x$

Change in volume of sediment in box:     $\Delta V = \Delta h\Delta s = \Delta z\Delta x = q_{v_{in}} \Delta t - q_{v_{out}} \Delta t$

$$\frac{\Delta h}{\Delta t} = -\frac{q_{v_{out}} - q_{v_{in}}}{\Delta s} ; \quad \frac{\Delta z}{\Delta t} = -\frac{q_{v_{out}} - q_{v_{in}}}{\Delta x} ; \quad \frac{\partial z}{\partial t} = -\frac{\partial q_v}{\partial x}$$

## E. Soil Mantled Slopes: Steady State Forms

### Generic Transport Relationship

Kirkby (1971):     $q_v = kq^m S^n = k'x^m S^n$

Kirkby gives empirical evidence for m,n values for different hillslope processes.

We will (later) examine evidence and derive values from theory, develop understanding of geologic, climatic and biotic control of various parameters. First we examine implications of different forms of the transport relationship for equilibrium hillslope form. Lab exercise will pursue numerical implementation to allow investigation of boundary conditions (link to rest of landscape) and hillslope responses to transients (e.g. change in climate or river incision rate)

Coupled with continuity equation - can derive relationships for steady-state slope forms developed under different sets of processes (e.g. different climates)

### Soil Creep

Generally Humid/temperate conditions:  $I_c \gg R_i$ ;  $m=0$ ,  $n=1$

Transport law:     $q_v = K_c S = K_c \left( -\frac{\partial z}{\partial x} \right)$

$K_c = f(\text{rainfall, windiness (tree throw), freeze-thaw cycles, soil texture, clay mineralogy, etc}) [L^2/T]$

Continuity: 
$$\frac{\partial z}{\partial t} = -\frac{\partial q_v}{\partial x}$$

Substitute: Diffusion Eqn 
$$\frac{\partial z}{\partial t} = -\frac{\partial}{\partial x} \left( -K_c \frac{\partial z}{\partial x} \right) = K_c \frac{\partial^2 z}{\partial x^2}$$

Uniform Erosion condition: 
$$\frac{\partial z}{\partial t} = -\dot{\epsilon} = K_c \frac{\partial^2 z}{\partial x^2}$$

Integrate (w/ respect to x): 
$$-\dot{\epsilon}x = K_c \frac{\partial z}{\partial x} + C_1$$

B.C.: no flux across ridge:  $S=0$  at  $x=0$   $C_1 = 0$  ;  $-\dot{\epsilon}x = K_c \frac{\partial z}{\partial x}$

Note this is solution for slope with uniform erosion: 
$$\frac{\partial z}{\partial x} = \frac{-\dot{\epsilon}}{K_c} x$$

Separate variables, integrate 
$$-\frac{1}{2} \dot{\epsilon} x^2 = K_c z + C_2$$

B.C.:  $z=z_0$  at ridge top,  $x=0$   $C_2 = -K_c z_0$

Uniform erosion solution (parabola) 
$$-\frac{1}{2} \dot{\epsilon} x^2 = K_c z - K_c z_0$$

$$z = z_0 - \frac{\dot{\epsilon} x^2}{2K_c}$$

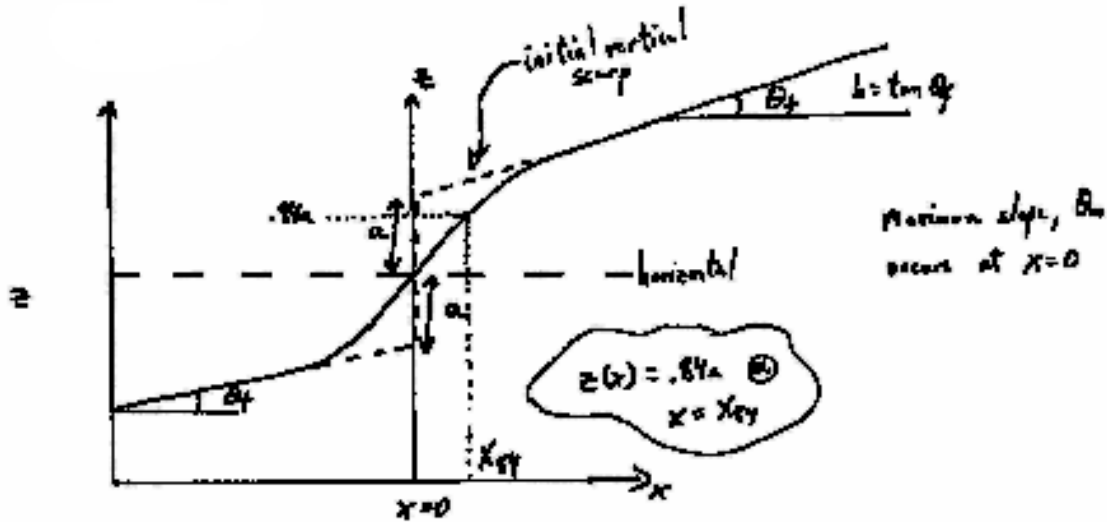
How are boundary conditions reflected in this formulation? Geologic and climatic factors?

### Use in Fault-Scarp Age Determination

Analytical solution for transient behavior: error function solution. Initial profiles assumed to be at angle of repose.

Diffusion models “fit” to observed slope profiles -- “fit” only derives estimate of  $K^*t$  product -- need independent estimate of  $t$  to “calibrate”  $K$  for field setting.

Sketch: Hanks et al, 1984 definition sketch.



Initial Condition: Vertical Scarp.

$$z(x,t) = a \operatorname{erf} \left[ \frac{x}{2\sqrt{Kt}} \right] + bx$$

Where  $a$  is scarp amplitude,  $b$  is far-field slope, and  $\operatorname{erf}$  is the error function:

$$\operatorname{erf} \left[ \frac{x}{2\sqrt{Kt}} \right] = \frac{2}{\sqrt{\pi}} \int_0^{x/(2\sqrt{Kt})} e^{-\eta^2} d\eta$$

Maximum scarp angle ( $\theta_m$ ) occurs at  $x = 0$  and has the convenient formulation:

$$\left[ \frac{dz}{dx} \right]_{x=0} = \tan \theta_m = \frac{a}{\sqrt{\pi Kt}} + b$$

If scarp cuts a flat surface,  $b = 0$ . Another convenient relation is found by recognizing that  $\operatorname{erf}(1) = 0.84$ . Thus if we define  $X_{84}$  as the position at which 84% of scarp offset  $a$  is reached, we can write:

$$X_{84} = 2\sqrt{Kt} \quad \text{or} \quad t = \frac{X_{84}^2}{4K}$$

$$K = \frac{X_{84}^2}{4t}$$

Pierce and Colman (1986) – Paper available – present a solution for a similar problem, with these differences: no background slope ( $b = 0$ ); initial condition = scarp angle  $\alpha$  (not a vertical scarp, usually the angle of repose, ca.  $33^\circ$ ). They solve for diffusivity,  $K$ , in terms of scarp age ( $t$ ) and morphologic measurement of

the maximum scarp gradient,  $\tan \theta_m$ , for a scarp of total height ( $h$ ) (not to be confused with the amplitude =  $\frac{1}{2}$  height used by Hanks et al):

$$K_c = \left[ \frac{h}{4t^{1/2} \operatorname{erf}^{-1}(\tan \theta_m / \tan \alpha) \tan \alpha} \right]^2$$

Solve for maximum slope as a function of time:

$$\operatorname{erf}^{-1}(\tan \theta_m / \tan \alpha) = \left[ \frac{h}{4(K_c t)^{1/2} \tan \alpha} \right]$$

$$(\tan \theta_m / \tan \alpha) = \operatorname{erf} \left[ \frac{h}{4(K_c t)^{1/2} \tan \alpha} \right]$$

$$\tan \theta_m = (\tan \alpha) \operatorname{erf} \left[ \frac{h}{4(K_c t)^{1/2} \tan \alpha} \right]$$

We will apply this relationship to calculate how the maximum slope of the cone (the straight part or base of convex region) would vary through time according to a linear soil creep model (acknowledging that the difference between a planar fault scarp and a cinder cone profile will affect the rate at which this maximum slope changes for a given  $K_c$ ).

Scarp studies (southwestern US: semi-arid; granular soils; sparse vegetation) generally find:  $10^{-3} < K_c < 10^{-2} \text{ m}^2/\text{yr}$ .

Dietrich et al. in humid/temperate N. California and Oregon coastal mountains (moist winter conditions, dense vegetation) find:  $K_c \sim 5 \times 10^{-3} \text{ m}^2/\text{yr}$ .

Small et al. (Geomorphology, 1999) on frost-dominated, unvegetated summit flats eroding at rates of  $\sim 15 \text{ } \mu\text{m}/\text{yr}$  estimate  $K_c \sim 1.7 \times 10^{-2} \text{ m}^2/\text{yr}$ .