Conservation of Mass (Rock): Profile Evolution

Now we can consider conservation of mass of the rock to write an evolution equation for a bedrock channel:

$$\frac{\partial z}{\partial t} = U - E = U - KA^m S^n$$

At **steady state**, $U = E$ such that $\frac{\partial z}{\partial t} = 0$, such that we can write:

$$U = KA^m S^n$$

this can be solved directly for the steady-state river slope:

$$S = \left(\frac{U}{K}\right)^{\frac{1}{n}} A^{-\frac{m}{n}}$$

Thus a power-law relation between local channel slope and upstream drainage area is predicted: a straight line in plot of log$S$ vs. log$A$ with slope $-m/n$ (concavity index) and intercept $(U/K)^{1/n}$ (steepness index) (this is true only if U, K, m, and n are spatially uniform … what might happen to profile concavity where $K = K(q_s) = K(x)$?)

Steady-State Longitudinal Channel Profile

By integrating the above relation we can derive an equation for the longitudinal profile of the river at steady state:

$$-\frac{\partial z}{\partial x} = S = \left(\frac{U}{K}\right)^{\frac{1}{n}} A^{-\frac{m}{n}}$$

To integrate we need to write A in terms of along-stream distance, x. A robust empirical relation known as Hack's law (Hack, 1957) allows this:

$A = k_a x^h$; where ka ~ 6.7 and h ~ 1.67 are typical values.

Substituting in and setting up to integrate both sides:

$$\int \frac{\partial z}{\partial x} = -\int \left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} x^{-\frac{hm}{n}}$$

$$z(x) = -\left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} \left(1 - \frac{hm}{n}\right)^{-1} x^{\left(1-\frac{hm}{n}\right)} + C \quad ; \quad \frac{hm}{n} \neq 1$$

$$z(x) = -\left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} \ln(x) + C \quad ; \quad \frac{hm}{n} = 1$$

To find constant of integration, set baselevel $z = z(L)$ at $x = L$

$$z(L) = -\left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} \left(1 - \frac{hm}{n}\right)^{-1} L^{\left(1-\frac{hm}{n}\right)} + C$$

$$z(x) = \left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} \left(1 - \frac{hm}{n}\right)^{-1} \left[L^{\left(1-\frac{hm}{n}\right)} - x^{\left(1-\frac{hm}{n}\right)}\right] + z(L) \quad ; \quad \frac{hm}{n} \neq 1 \quad ; \quad x_c \leq x \leq L$$

$$z(x) = \left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} \left[\ln(L) - \ln(x)\right] + z(L) \quad ; \quad \frac{hm}{n} = 1 \quad ; \quad x_c \leq x \leq L$$

where x_c (~ 200-1000m typical) is the distance from the divide at which fluvial processes become dominant over hillslope processes (soil creep, landslides, etc) and debris flow scour.

Fluvial Relief of Drainage Basins

Fluvial Relief is thus given by:

$$R_f = z(x_c) - z(L) = \left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} \left(1 - \frac{hm}{n}\right)^{-1} \left[L^{\left(1-\frac{hm}{n}\right)} - x_c^{\left(1-\frac{hm}{n}\right)}\right] \quad ; \quad \frac{hm}{n} \neq 1$$

$$R_f = z(x_c) - z(L) = \left(\frac{U}{K}\right)^{\frac{1}{n}} k_a^{-\frac{m}{n}} \left[\ln(L) - \ln(x_c)\right] \quad ; \quad \frac{hm}{n} = 1$$

Note that all except U, K are geometrical variables, so convenient to simplify to:

$$R_f = z(x_c) - z(L) = \beta\left(\frac{U}{K}\right)^{\frac{1}{n}}$$

where β is expected to vary weakly with tectonic, lithologic, and climatic conditions and is given by:

$$\beta = k_a^{-\frac{m}{n}} \left(1 - \frac{hm}{n}\right)^{-1} \left[L^{\left(1-\frac{hm}{n}\right)} - x_c^{\left(1-\frac{hm}{n}\right)} \right] \quad ; \quad \frac{hm}{n} \neq 1$$

$$\beta = k_a^{-\frac{m}{n}} \left[\ln(L) - \ln(x_c)\right] \quad ; \quad \frac{hm}{n} = 1$$

Channel Profiles and Fluvial Relief – Empirical Geometric Constraints

The above all derived for steady-state conditions where bedrock channel incision is described by the stream power model and *U*, *K*, *m*, and *n* are uniform in space (same tectonics, climate, lithology, and erosion process) – a fairly restrictive set of assumptions. However, it is commonly observed that river profiles follow a power-law relation between channel gradient and upstream drainage area:

$$S = k_s A^{-\theta}$$

where k_s is the steepness index and θ is the concavity index.

Thus the above derivations for profile form and fluvial relief can be repeated for channels with this empirically observed form, yielding equivalent relations that are not directly tied to the above list of assumptions, ie. these relations are valid even if the stream power river incision model is not:

$$z(x) = k_s k_a^{-\theta} (1 - h\theta)^{-1} \left[L^{(1-h\theta)} - x^{(1-h\theta)} \right] + z(L) \quad ; \quad h\theta \neq 1 \quad ; \quad x_c \leq x \leq L$$

$$z(x) = k_s k_a^{-\theta} \left[\ln(L) - \ln(x)\right] + z(L) \quad ; \quad h\theta = 1 \quad ; \quad x_c \leq x \leq L$$

$$R_f = z(x_c) - z(L) = \beta k_s$$

$$\beta = k_a^{-\theta} (1 - h\theta)^{-1} \left[L^{(1-h\theta)} - x_c^{(1-h\theta)} \right] \quad ; \quad h\theta \neq 1$$

$$\beta = k_a^{-\theta} \left[\ln(L) - \ln(x_c)\right] \quad ; \quad h\theta = 1$$

C. River Incision Processes

Powerpoint Presentation: Physical Erosion Processes; plus lecture on constraints on how erosion processes scale with mean bed shear stress (ie. what is exponent *a* for different processes?)

Topics Discussed During Presentation

River Incision into Bedrock:

- Interaction of a suite of process
    - Plucking, Abrasion (bedrock & susp load), Cavitation (?), Weathering
- Vortices shed off macro-roughness drive processes
    - Relation to mean bed shear stress?
- Critical stress for incision/flood frequency
- Adjustment of channel morphology/bed state
- How non-linear? Relation to Climate?

D. Weaknesses of the Stream Power River Incision Model

- Neglects critical shear stress for incision (assumed exceeded in floods of interest)
- Exponent *a* and k_b unknown and depend on process(es) active -- model is not specific about process mechanics, just “scales with shear stress”
- k_b should depend on sediment flux – details uncertain, nothing explicit in model
- assumes steady, uniform flow, but much erosion may be related to knickpoints and local flow accelerations – at what scale should *S* be measured?
- Roughness assumed constant in space (and with flow depth)
- No model for channel width, just assumed to follow hydraulic geometry
- No explicit treatment of flood frequency
- Basin hydrology ($Q \sim A^c$) best for moderate floods. Extreme events can be point-source events
- Small angle approximation breaks down in steep mountain channels and on knickpoints (but minor in comparison to other concerns)

E. Transient Profile Form and Landscape Response Time (time to steady state)

Consider response of landscape starting at an initial steady state and subjected to a sudden step-function change in either rock uplift rate (U) or climate (K). Transient profile consists of two sections separated by an abrupt change in slope – a knickpoint. Downstream of the knickpoint the channel profile is adjusted to the new conditions (steady state with U_f and/or K_f); upstream of the knickpoint the channel profile reflects the old steady-state conditions (steady state with U_i and/or K_i).

The profile reaches steady state when the lower segment reaches $x = x_c$, or when:

$$z(x_c) = z_f(x_c)$$

Time to steady state is given by the ratio of the total change in elevation at $x = x_c$ to the rate of change of elevation at $x = x_c$:

$$T = \frac{distance}{velocity} = \frac{z_f(x_c) - z_i(x_c)}{\partial z(x_c)/\partial t}$$

We have from above that at steady state:

$$z_i(x_c) = \beta\left(\frac{U_i}{K_i}\right)^{\frac{1}{n}} \quad ; \quad z_f(x_c) = \beta\left(\frac{U_f}{K_f}\right)^{\frac{1}{n}}$$

$$\beta = k_a^{-\frac{m}{n}}\left(1 - \frac{hm}{n}\right)^{-1}\left[L^{\left(1-\frac{hm}{n}\right)} - x_c^{\left(1-\frac{hm}{n}\right)}\right] \quad ; \quad \frac{hm}{n} \neq 1$$

$$\beta = k_a^{-\frac{m}{n}}\left[\ln(L) - \ln(x_c)\right] \quad ; \quad \frac{hm}{n} = 1$$

Further, we can deduce from the transient profile form that:

$$\partial z(x_c)/\partial t = const = U_f - E_i(K_f/K_i) = U_f - U_i(K_f/K_i)$$

for a change in U only, $K_f/K_i = 1$; for a change in K only, $U_f = U_i = U$

Thus, defining the fractional change in uplift and the coefficient of erosion as:

$$f_U = U_f/U_i \quad ; \quad f_K = K_f/K_i$$

we may write the rate of change of elevation at $x = x_c$ as:

$$\partial z(x_c)/\partial t = U_i(f_U - 1) \quad \text{for a change in } U \text{ only}$$

$$\partial z(x_c)/\partial t = U(1 - f_K) \quad \text{for a change in } K \text{ only}$$

Thus system response time is given simply by:

$$T_U = \frac{z_f(x_c) - z_i(x_c)}{\partial z(x_c)/\partial t} = \frac{\beta K^{\frac{-1}{n}} U_i^{\frac{1}{n}-1}\left(f_U^{\frac{1}{n}} - 1\right)}{\left(f_U - 1\right)}$$

$$T_K = \frac{z_f(x_c) - z_i(x_c)}{\partial z(x_c)/\partial t} = \frac{\beta K_i^{\frac{-1}{n}} U^{\frac{1}{n}-1}\left(f_K^{\frac{-1}{n}} - 1\right)}{\left(1 - f_K\right)}$$

Assumptions:

- $x_c \neq f(U, K)$; $L >> x_c$ → β = constant
- $S_i = k_{s_i} A^{-\theta}$, $S_f = k_{s_f} A^{-\theta}$ (concavity invariant, k_s function of uplift rate)
    - For stream power model, $k_{s_i} = (\frac{U}{K_i})^{\frac{1}{n}}$, $k_{s_f} = (\frac{U}{K_f})^{\frac{1}{n}}$
- Slope is unchanged above knickpoint

  Retain sharp knickpoint $\Rightarrow$ no information is passed upstream

  T_U : 1Ma (order of)

Vertical Knickpoint Velocity

Objective: Use the solution for response time above to solve for vertical knickpoint velocity. Key: knickpoint travels (in *z*) from the basin outlet to the final position of the fluvial channel head … over the full distance of the new steady-state fluvial relief, in the same total amount of time.

Set this definition of response time equal to the one derived above (they are two ways to express the same thing):

$$T_U = \frac{\Delta z_{\text{knick}}}{V_{kp}} = \frac{z(x_c)_f}{V_{kp}} = \frac{\beta K^{-\frac{1}{n}}(U_f^{\frac{1}{n}} - U_i^{\frac{1}{n}})}{U_f - U_i}$$

$$z(x_c)_f = \beta K^{-\frac{1}{n}} U_f^{\frac{1}{n}}$$

Solve for knickpoint velocity:

$$V_{kp} = \frac{U_f^{\frac{1}{n}}(U_f - U_i)}{U_f^{\frac{1}{n}} - U_i^{\frac{1}{n}}} = \frac{f_U^{\frac{1}{n}} U_i (f_U - 1)}{f_U^{\frac{1}{n}} - 1}$$

where $f_U = U_f / U_i$

thus, $V_{kp} = U_f$ when $n = 1$

- transient, U goes up, K goes down $\Rightarrow$ knickpoint moves upstream at constant vertical rate (all lie on the same contour within a basin!)
- Δ lithology $\Rightarrow$ fixed knickpoint
- Δ uplift (across a fault) $\Rightarrow$ fixed knickpoint